DRAFT: Analysis of Skew Tensegrity Prisms

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Abstract

This paper describes the properties of skew tensegrity prisms. By showing that the analytical equilibrium solutions of regular tensegrity prisms still hold for their skewed counterparts – making use of the fact that the equilibrium of a tensegrity structure is preserved under an affine transformation – it validates and explains previous numerical results.

1 Introduction

Tensegrity prisms have regularly been analysed in tensegrity literature, as they allow for analytical equilibrium solutions (e.g. Connelly and Terrell, 1995; Hinrichs, 1984; Murakami, 2001). Tensegrity prisms are here taken to be tensegrities where the vertices form two equilateral polygons on parallel planes (where all vertices in a plane lie on a circle). For regular tensegrity prisms the perpendicular through the centre of one end of the prism is parallel to, and coincident with, a perpendicular through the centre of the other end. For a prism to be strictly skew, the perpendiculars are still parallel, but are not coincident (Burkhardt, 2006).

Burkhardt (2006) describes a non-linear programming approach to designing skew tensegrity prisms, and several observations are made. This paper explains the observation that the twist angle is the same for both cases, and shows that the analytical equilibrium solutions from the regular tensegrity prisms also apply to their skewed counterparts.

2 Tensegrity equilibrium

2.1 Analytical solutions

The analytical equilibrium solutions for rotationally symmetric tensegrity prisms have been derived before. Connelly and Terrell (1995) derived the equilibrium equations for a wide variety of connectivity combinations, using the terminology of Hinrichs (1984), but for equal bottom and top radius of the prisms. In this case the simplest connectivity type is chosen, but with differing top and bottom radius r_h and r_0 (see Fig. 1).

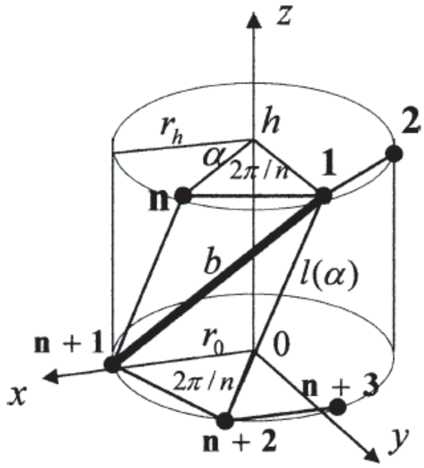

Figure 1: Rotationally symmetric tensegrity prisms, with n-polygons on two parallel planes, twisted over angle α with respect to eachother.

The equilibrium is given in terms in the tension coefficients $\hat{t} = \frac{t}{l}$, as the ratio of the tension in, and length of the member.

$$\hat{t}_v = -\hat{t}_b \tag{1}$$

$$\hat{t}_b = -2\frac{r_h}{r_0}\sin\left(\frac{\pi}{n}\right)\hat{t}_h \tag{2}$$

$$\frac{\hat{t}_h}{\hat{t}_0} = \left(\frac{r_0}{r_h}\right)^2 \tag{3}$$

The twist angle α is independent of either radius or height, and is given by the number of nodes of the polygons:

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n}. \tag{4}$$

These equations allow the analytical solutions to the equilibrium equations of the regular prism tensegrities.

2.2 Numerical solutions

Most tensegrity structures do not have analytical solutions, and several numerical methods for finding the initial equilibrium configurations, also referred to as *form-finding*, have been developed. As observed by Tibert and Pellegrino (2003) the *energy method* from mathematical rigidity theory is equivalent to the engineering *force density method* (e.g. Vassart and Motro, 1999). Here we will be using the terminology and notations of rigidity theory to describe the equilibrium of a tensegrity.

Literature in rigidity theory makes use of the stress matrix $\tilde{\mathbf{\Omega}}$ to describe the equilibrium of a tensegrity structure. It relates the external forces to the internal tensions, and is created by writing the force equilibrium at each of the nodes

$$\sum_j \omega_{ij} \left(\mathbf{p}_j - \mathbf{p}_i \right) = \mathbf{0} \tag{5}$$

where $\mathbf{p}_i$ are the coordinates for node i, and ω_{ij} is the tension in the member connecting nodes i and j, divided by the length of the member; ω_{ij} is referred to as a *stress* in rigidity theory, but is known in engineering as a *force density* or *tension coefficient*. If all the nodal coordinates are written together as a vector $\mathbf{p}$, $\mathbf{p}^T = [\mathbf{p}_1^T, \mathbf{p}_2^T, \ldots, \mathbf{p}_n^T]$, the equilibrium equations at each node can be combined to obtain the matrix equation

$$\tilde{\mathbf{\Omega}} \mathbf{p} = \mathbf{0}. \tag{6}$$

Because the equilibrium is the same for each of the d dimensions, the stress matrix can be written as the Kronecker product of the reduced stress matrix $\mathbf{\Omega}$ and a d-dimensional identity matrix $\mathbf{I}^d$:

$$\tilde{\mathbf{\Omega}} = \mathbf{\Omega} \otimes \mathbf{I}^d. \tag{7}$$

Using the tension coefficients ω of each of the members the coefficients of the reduced stress matrix are then given as

$$\Omega_{ij} = \begin{cases} -\omega_{ij} = -\omega_{ji} & \text{if } i \neq j \text{, and \{i,j\} a member,} \\ \sum_{k \neq i} \omega_{ik} & \text{if } i = j, \\ 0 & \text{if there is no connection between } i \text{ and } j. \end{cases} \tag{8}$$

Note that the stress matrix merely contains information about element connectivity and tension coefficients, but no information about the geometry of the structure.

We have presented the two approaches to describing the equilibrium of a tensegrity structure: analytically and numerically. In the next section it will be shown that by choosing to define the equilibrium in terms of the tension coefficients for each of the methods, interesting conclusions can be made.

3 Affine transformations

Literature has shown that the equilibrium of a tensegrity structure is preserved under an affine transformation (e.g. Connelly and Back, 1998; Masic et al., 2005). Affine transformations are linear transformations of coordinates (of the whole affine plane onto itself) preserving collinearity. Thus, an affine transformation transforms parallel lines into parallel lines and preserves ratios of distances along parallel lines, as well as intermediacy (Coxeter, 1989, pp. 202). We write them as the transformation of the coordinates of node i

$$\mathbf{p}_i \rightarrow \mathbf{U}\mathbf{p}_i + \mathbf{w}$$

where in d-dimensional space $\mathbf{U}$ is a d-by-d matrix, and $\mathbf{w} \in E^d$. This provides a total of $d(d+1)$ independent affine transformations.

Affine transformations are well-known to engineers, but under a different guise. Recall that the square matrix $\mathbf{U}$ can be expressed as the sum of a symmetric and a skew-symmetric component. Then the half of the $d(d+1)$ affine transformations constituted by $\mathbf{w}$ and the skew-symmetric part of $\mathbf{U}$, is better known to engineers as rigid-body motions (e.g. 6 rigid-body motions in 3-dimensional space). The interpretation of the other half – the symmetric part of $\mathbf{U}$ – is less obvious, but it turns out to be equivalent to the basic strains found in continuum mechanics: shear and dilation. For instance, for a 3-dimensional strain, infinitesimal affine deformations give the six independent strain quantities (e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{xz}, e_{yz}).

In order to show that equilibrium is preserved under an affine transformation, we look at equation 5. When considering that

$$\begin{aligned} \mathbf{p}_i &\rightarrow \mathbf{U}\mathbf{p}_i + \mathbf{w}, \\ \mathbf{p}_j &\rightarrow \mathbf{U}\mathbf{p}_j + \mathbf{w}, \end{aligned}$$

then

$$\begin{aligned} \sum_j \omega_{ij} \left(\mathbf{U}\mathbf{p}_j + \mathbf{w} - \mathbf{U}\mathbf{p}_i - \mathbf{w} \right) &= \mathbf{0} \\ \sum_j \omega_{ij} \left(\mathbf{U}\mathbf{p}_j - \mathbf{U}\mathbf{p}_i \right) &= \mathbf{0} \\ \mathbf{U} \sum_j \omega_{ij} \left(\mathbf{p}_j - \mathbf{p}_i \right) &= \mathbf{0}. \end{aligned}$$

This means that under an affine transformation the equilibrium condition still holds, and as a result the stress matrix is invariant. It is obvious that the equilibrium would hold for rigid-body motions, but not for the other affine transformations. The subsequent observation that affine transformations are in the nullspace of the stress matrix, has been used as basis for describing tensegrities with zero stiffness (Schenk et al., 2006).

The observation that the stress matrix remains invariant under an affine transformation further means that the tension coefficients for each of the members remains constant. When we realize that the skewed tensegrity prisms of Burkhardt (2006) are merely an affine transformation of the regular tensegrity prisms, it is clear that the analytical values for the tension coefficients still apply those skewed versions! Furthermore, for a pure skew transformation where top and bottom polygon remain planar and parallel, the twist angle will remain unchanged. This explains the observation made by Burkhardt.

4 Example structures

Burkhardt (2006) describes two skew tensegrity prisms, using a non-linear programming method. In these cases the affine transformation is purely a skew between bottom and top plane of the prism. As a result the twist angle calculated earlier remains valid: the relative rotation cannot change when the planes remain parallel.

4.1 Three bar prism

The dataset in Table 1 is copied from Burkhardt, but the column with tension coefficients is added. The radius of top and bottom polygon is identical and is equal to $\frac{1}{\sqrt{3}} = 0.577$. Using Eqn. 3 we can now verify the data:

$$\hat{t}_b = -2\frac{r_h}{r_0}\sin\left(\frac{\pi}{n}\right)\hat{t}_h = -2\sin\left(\frac{\pi}{3}\right)\hat{t}_h = -\sqrt{3}\hat{t}_h \quad (9)$$

$$\hat{t}_v = -\hat{t}_b \quad (10)$$

$$\hat{t}_h = \hat{t}_0 \quad (11)$$

By taking $\hat{t}_b = -1$, we find $\hat{t}_v = 1$ and $\hat{t}_h = \frac{1}{\sqrt{3}} = 0.577$. These values correspond to the numerical values from Burkhardt.

Table 1: Dataset from Burkhardt for three-bar skew tensegrity prism, with additional column for tension coefficients.

Member	Tension	Length	Tension Coefficient
1	0,577	1,000	0,577
2	0,577	1,000	0,577
3	0,577	1,000	0,577
4	1,300	1,300	1,000
5	1,579	1,579	1,000
6	1,272	1,272	1,000
7	-2,066	2,066	-1,000
8	-2,000	2,000	-1,000
9	-1,000	1,000	-1,000
10	0,577	1,000	0,577
11	0,577	1,000	0,577
12	0,577	1,000	0,577

4.2 Four bar prism

Again, the dataset in Table 2 is copied from Burkhardt, but the column with tension coefficients is added. Using the dataset, we can determine top radius $r_h = 0.64$ and bottom radius $r_0 = 0.71$. Using Eqn. 3 we can now verify the data:

$$\hat{t}_b = -2\frac{r_h}{r_0}\sin\left(\frac{\pi}{n}\right)\hat{t}_h = -2\frac{0.64}{0.71}\sin\left(\frac{\pi}{4}\right)\hat{t}_h = -1.275\hat{t}_h \quad (12)$$

$$\hat{t}_v = -\hat{t}_b \quad (13)$$

$$\frac{\hat{t}_h}{\hat{t}_0} = \left(\frac{r_0}{r_h}\right)^2 = \left(\frac{0.71}{0.64}\right)^2 = 1.232 \quad (14)$$

By taking $\hat{t}_b = -1$, we find $\hat{t}_v = 1$ and $\hat{t}_h = \frac{1}{1.275} = 0.785$ and $\hat{t}_0 = \frac{0.784}{1.232} = 0.637$. These values correspond to the numerical values from Burkhardt.

Table 2: Dataset from Burkhardt for four-bar skew tensegrity prism, with additional column for tension coefficients.

Member	Tension	Length	Tension Coefficient
1	0,707	0,901	0,785
2	0,707	0,901	0,785
3	0,707	0,901	0,785
4	0,707	0,901	0,785
5	1,700	1,700	1,000
6	2,053	2,053	1,000
7	1,348	1,348	1,000
8	1,772	1,772	1,000
9	-2,516	2,516	-1,000
10	-1,500	1,500	-1,000
11	-2,516	2,516	-1,000
12	-1,500	1,500	-1,000
13	0,637	1,000	0,637
14	0,637	1,000	0,637
15	0,637	1,000	0,637
16	0,637	1,000	0,637

5 Conclusions

By combining the analytical solutions for the regular tensegrity prisms with the knowlede of affine transformations, we can describe and understand the skew tensegrity prisms much better. Under a pure skew transformation, preserving parallel planes, the twist angle must remain constant. This verifies the observation of Burkhardt (2006). Furthermore, the analytical equilibrium solutions for the regular tensegrity prisms can be applied to any skewed version, including where the top and bottom polygon are no longer equilateral.

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